A CONDITION FOR UNIVALENCE IN THE POLYDISC

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Abstract. We study a sufficient condition for univalence in the polydisk in terms of the size of the norm of the Schwarzian operator. Examples show that our result is close to optimal in dimension two. This paper extends work by the second author concerning similar criteria in the ball.

1. Introduction

The Schwarzian derivative

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of a locally injective analytic map f has been studied extensively in one complex variable, especially in connection with necessary and sufficient conditions for univalence on domains. It is invariant under compostions $T \circ f$ with Möbius transformations T, which are the only mappings that have Schwarzian vanishing everywhere. The associated linear equation $u'' + \frac{1}{2}(Sf)u = 0$ plays an important role since f is univalent in a simply-connected domain Ω if and only if every non-trivial solution u of the linear equation vanishes in Ω at most once. This is a consequence of the fact that any mapping f with $Sf = 2p$ is given as $f = u_1/u_2$ for two linearly independent solutions of $u + pu = 0$. Under suitable bounds for $|Sf|$, variants of Sturm comparison techniques allow then to preclude multiple zeros of u. We cite the pioneer work of Nehari [5], who among other criteria proved that if f is analytic, locally univalent in $\mathbb{D} = \{z : |z| < 1\}$ and

$$(1 - |z|^2)^2 |Sf(z)| \leq 2\,,$$

then f is univalent. This important class of univalent mappings contains, for example, all convex functions (see [6]). In 1972, J. Becker gave a criterion using the pre-Schwarzian f''/f', namely that

$$(1 - |z|^2)\left|z\frac{f''}{f'}(z)\right| \leq 1$$

The authors were partially supported by Fondecyt Grants # 1071019 and # 11070055.
Key words: Schwarzian operator, univalence condition, polydisk, Bergman metric.
2000 AMS Subject Classification. Primary: 32H02, 32A17; Secondary: 30C45.

implies that f is univalent and $\Omega = f(\mathbb{D})$ a Jordan domain, [1]. The constants 2 and 1 in both results are sharp.

The purpose of this paper is to study a similar sufficient condition for the univalence of a locally biholomorphic mapping defined in the polydisk. We will employ a generalization of the Schwarzian derivative developed in [2] and based on work by T.Oda in [7]. In several variables there is a family of Schwarzian derivatives $S_{ij}^k F$ associated with a single mapping F, which can be used to define a Schwarzian operator $\mathcal{S}F$ that inherits a norm $||\mathcal{S}F||$ from any hermitian norm defined on the domain. Our result, close to optimal in dimension $n = 2$ and less so in higher dimensions, constitutes a complement of the work in [3], where the second author establishes sufficient conditions for univalence in the ball in terms of suitable bounds for the norm $||\mathcal{S}F||$ relative to the Bergman metric. This paper represents a step toward generalizing the classical theme of univalence criteria to several variables, and highlights, in yet another way, differences between the ball and the polidisk.

2. The Schwarzian

Let $F : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping defined on some domain Ω. T.Oda in [7] defined a family of Schwarzian derivatives of $F = (f_1, \ldots, f_n)$ as

$$S_{ij}^k F = \sum_{l=1}^{n} \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1} \left(\delta_i^k \frac{\partial}{\partial z_j} + \delta_j^k \frac{\partial}{\partial z_i} \right) \log \Delta \,, \tag{2.1}$$

where $i, j, k = 1, 2, \ldots, n$, $\Delta = \det(DF)$ is the jacobian determinant of the diferential DF and δ_i^k are the Kronecker symbols. For $n > 1$ the Schwarzian derivatives have the following properties:

$$S_{ij}^k F = 0 \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n \quad \text{iff} \quad F(z) = M(z) \,, \tag{2.2}$$

for some Möbius transformation

$$M(z) = \left(\frac{l_1(z)}{l_0(z)}, \ldots, \frac{l_n(z)}{l_0(z)} \right) ,$$

where $l_i(z) = a_{i0} + a_{i1} z_1 + \cdots + a_{in} z_n$ with $\det(a_{ij}) \neq 0$. Furthermore, for a composition

$$S_{ij}^k (G \circ F)(z) = S_{ij}^k F(z) + \sum_{l,m,r=1}^{n} \mathbb{S}_{lm}^r G(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r} \,,\ w = F(z) \,. \tag{2.3}$$

Thus, if G is a Möbius transformation then $S_{ij}^k(G\circ F) = S_{ij}^k F$. The $S_{ij}^0 F$ coefficients are given by

$$S_{ij}^0 F(z) = \Delta^{1/(n+1)} \left(\frac{\partial^2}{\partial z_i \partial z_j} \Delta^{-1/(n+1)} - \sum_{k=1}^n \frac{\partial}{\partial z_k} \Delta^{-1/(n+1)} S_{ij}^k F(z) \right) .$$

In his work, Oda gives a description of the functions with prescribed Schwarzian derivatives $S_{ij}^k F$ ([7]). Consider the following overdetermined system of partial differential equations,

$$\frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^n P_{ij}^k(z) \frac{\partial u}{\partial z_k} + P_{ij}^0(z) u \ , \quad i,j = 1,2,\ldots,n \, , \tag{2.4}$$

where $z = (z_1, z_2, ..., z_n) \in \Omega$ and $P_{ij}^k(z)$ are holomorphic functions in Ω, for $i,j,k = 0,\ldots,n$. The system (2.4) is called *completely integrable* if there are $n+1$ (maximun) linearly independent solutions, and is said to be in *canonical form* (see [8]) if the coefficients satisfy

$$\sum_{j=1}^n P_{ij}^j(z) = 0 \ , \quad i = 1,2,\ldots,n.$$

T. Oda proved that (2.4) is a completely integrable system in canonical form if and only if $P_{ij}^k = S_{ij}^k F$ for a locally boholomorphic mapping $F = (f_1,\ldots,f_n)$, where $f_i = u_i/u_0$ for $1 \le i \le n$ and $u_0, u_1, \ldots, u_n$ is a set of linearly independent solutions of the system. For a given mapping F, $u_0 = (\Delta)^{-\frac{1}{n+1}}$ is always a solution of (2.4) with $P_{ij}^k = S_{ij}^k F$.

We recall the following definitions from [2], where the individual Schwarzians $S_{ij}^k F$ are grouped adequately as an operator.

Definition 2.1. For each $k = 1,\ldots,n$ we let $\mathbb{S}^k F$ be the $n \times n$ matrix

$$\mathbb{S}^k F = (S_{ij}^k F) \, , \quad i,j = 1,\ldots,n \, .$$

Definition 2.2. We define the *Schwarzian derivative operator* as the mapping $\mathcal{S}F(z) : T_z\Omega \to T_{F(z)}\Omega$ given by

$$\mathcal{S}F(z)(\vec{v}) = \left(\vec{v}^t \mathbb{S}^1 F(z) \vec{v} \, , \, \vec{v}^t \mathbb{S}^2 F(z) \vec{v} \, , \, \ldots , \vec{v}^t \mathbb{S}^n F(z) \vec{v} \right) \, ,$$

where $\vec{v} \in T_z\Omega$.

The Bergman metric on the polydisk $\mathbb{P}^n$ is the hermitian product defined by the diagonal matrix

$$g_{ii}(z) = \frac{2}{(1-|z_i|^2)^2} \, , \tag{2.5}$$

see, e.g., [4]. Is well known that the automorphisms group of the polydisc, up to multiplication by a diagonal unitary transformation and a permutation of the coordinates, consists of mappings

$$\psi(z) = \psi_a(z) = (\psi_{a_1}(z_1), \ldots, \psi_{a_n}(z_n))\,, \quad z = (z_1, \ldots, z_n) \in \mathbb{P}^n\,,$$

where $a = (a_1, \ldots, a_n) \in \mathbb{P}^n$ and $\psi_{a_j}(z_j) = \dfrac{z_j - a_j}{1 - \bar{a}_j z_j}$, $1 \le j \le n$. The polydisc is a homogeneous domain, however the action of this group is not transitive on the set of directions at a given point. We define the norm of the Schwarzian derivative operator by

$$\|\mathcal{S}F(z)\| = \sup_{\|\vec{v}\|=1} \|\mathcal{S}F(z)(\vec{v}\,)\|\,,$$

where

$$\|\,\vec{v}\,\| = \left[\,2\sum_{i=1}^{n} \frac{|v_i|^2}{(1-|z_i|^2)^2}\,\right]^{1/2}$$

is the Bergman norm of $\vec{v} \in T_z\mathbb{P}^n$. Finally, we let

$$||\mathcal{S}F|| = \sup_{z\in\mathbb{P}^n} ||\mathcal{S}F(z)||\,.$$

Because the automorphisms M of the ball $\mathbb{B}^n$ are Bergman isometries as well as Möbius, the corresponding norm $||\mathcal{S}F||$ remains invariant under composition $F \circ M$. Therefore, the class of mappings $F : \mathbb{B}^n \to \mathbb{C}^n$ for which $\|\mathcal{S}F\| \le \alpha$ is a linearly invariant family, and also normal after normalization (see [2]). The corresponding family in $\mathbb{P}^n$ fails to be linearly invariant.

3. Preliminary Lemmas

The following lemma is crucial in our work.

Lemma 3.1. *Let $F : \mathbb{P}^n \to \mathbb{C}^n$ be a locally univalent function with $\|\mathcal{S}F\| \le \alpha < \infty$, then*

$$|S_{ij}^k F(z)| \le \frac{\sqrt{2}\,\alpha\,(1-|z_k|^2)}{(1-|z_i|^2)(1-|z_j|^2)}\,.$$

Proof. Let $\vec{e_i} = (0, \ldots, 1, \ldots, 0)$ be the canonical vector in the i-th direction, and consider the unitary vector in the Bergman metric given by $\vec{u_i} = \frac{1}{\sqrt{2}}(1-|z_i|^2)\,\vec{e_i}$. Then

$$\|(\vec{u_j})^t\,\mathcal{S}F(z)\,\vec{u_i}\|^2 = \frac{1}{2}\sum_{k=1}^{n} \left|S_{ij}^k F(z)\right|^2 \frac{(1-|z_i|^2)^2(1-|z_j|^2)^2}{(1-|z_k|^2)^2} \le \alpha^2\,,$$

hence

$$|S_{ij}^k F(z)|^2 \le 2\alpha^2(1-|z_k|^2)^2(1-|z_i|^2)^{-2}(1-|z_j|^2)^{-2}\,,$$

as claimed. □

It follows from the lemma that, for $k \neq i, j$, $|S_{ij}^k F(z)| \to 0$ when $|z_k| \to 1$, and therefore by the maximum principle,

$$S_{ij}^k F(z) \equiv 0 \,, \quad k \neq i, j \,. \tag{3.1}$$

The vanishing of these Schwarzians for mappings with bounded $||\mathcal{S}F||$ is characteristic of the polydisk and does not occur in the ball.

Lemma 3.2. *Let $F = (f_1, \dots, f_n) : \mathbb{P}^n \to \mathbb{C}^n$ satisfy $||\mathcal{S}F|| \leq \alpha$, and suppose that $F(\xi) = 0$ and $DF(\xi) = I$ at some $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{P}^n$. Then for each i, the component f_i has a representation of the form*

$$f_i(z) = \sum_{n=1}^{\infty} a_{in}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)(z_i - \xi_i)^n \,, \tag{3.2}$$

where a_{in} is a holomorphic function independent of the variable z_i.

Proof. Let $u_0 = \Delta^{-\frac{1}{n+1}}$. Then $f_i u_0 = u_i$ where $u_0, u_1, \dots, u_n$ is a set of linearly independent solutions of (2.4) with $P_{ij}^k = S_{ij}^k F$. Differentiating $f_i u_0 = u_i$ with respect to z_j and z_k gives

$$\frac{\partial^2 f_i}{\partial z_j \partial z_k} u_0 + \frac{\partial f_i}{\partial z_j}\frac{\partial u_0}{\partial z_k} + \frac{\partial f_i}{\partial z_k}\frac{\partial u_0}{\partial z_j} + f_i \frac{\partial^2 u_0}{\partial z_j \partial z_k} = \frac{\partial^2 u_i}{\partial z_j \partial z_k} \,.$$

Because $S_{ij}^k F \equiv 0$ for each $k \neq i, j$, it follows from (2.4) that

$$\frac{\partial^2 f_i}{\partial z_j \partial z_k} u_0 + \frac{\partial f_i}{\partial z_j}\frac{\partial u_0}{\partial z_k} + \frac{\partial f_i}{\partial z_k}\frac{\partial u_0}{\partial z_j} = S_{jk}^j F \frac{\partial f_i}{\partial z_j} u_0 + S_{jk}^k F \frac{\partial f_i}{\partial z_k} u_0 \,. \tag{3.3}$$

By evaluating at $z = \xi$, we conclude that for $i \neq j, k$

$$\frac{\partial^2 f_i}{\partial z_j \partial z_k}(\xi) = 0 \,. \tag{3.4}$$

Moreover, from differentiating (3.3) with respect to z_l with $i \neq l$, we conclude after evaluating at $z = \xi$ that

$$\frac{\partial^3 f_i}{\partial z_j \partial z_k \partial z_l}(\xi) = 0 \,. \tag{3.5}$$

This procedure can be repeated to obtain for $i \neq k_1, \dots, k_m$

$$\frac{\partial^m f_i}{\partial z_{k_1} \cdots \partial z_{k_m}}(\xi) = 0 \,.$$

It follows that the Taylor series of f_i has the form

$$f_i(z) = \sum_{n=1}^{\infty} a_{in}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)(z_i - \xi_i)^n \,, \tag{3.6}$$

where a_{in} are holomorphic functions for each n.

□

The lemma implies that $f_i(z_1,\dots,z_{i-1},\xi_i,z_{i+1},\dots,z_n)=0$ for each $i=1,\dots,n$.

Lemma 3.3. *Let $F=(f_1,\dots,f_n):\mathbb{P}^n\to\mathbb{C}^n$ satisfy $||\mathcal{S}F||\le\alpha$, and suppose that $F(\xi)=0$ and $DF(\xi)=I$ at some $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{P}^n$. Then $\partial f_i/\partial z_i\neq 0$ in $\mathbb{P}^n$.*

Proof. To simplify notation, we will assume that $\xi=(0,\dots,0)$ even though the argument is valid for arbitrary ξ. We argue by contradiction, and assume that for some i the partial derivative $\partial f_i/\partial z_i=0$ at some point in $\mathbb{P}^n$. Without loss of generality, we make take $i=1$, that is, that $\partial f_1/\partial z_1(a)=0$ at some $a=(a_1,\dots,a_n)\in\mathbb{P}^n$. We will show that $\partial f_1/\partial z_i(0,a_2,\dots,a_n)=0$ for all $i=1,2,\dots,n$, leading to a contradiction with the fact that $\Delta\neq 0$ in $\mathbb{P}^n$. Equation (3.6) shows that

$$f_1(0,z_2,\dots,z_n)\equiv 0\,,$$

hence

$$\frac{\partial f_1}{\partial z_i}(0,a_2,\dots,a_n)=0 \quad , \; i=2,\dots,n\,.$$

Because $\Delta(a)\neq 0$ there exists $i\in\{2,\dots,n\}$ such that $(\partial f_1/\partial z_i)(a)\neq 0$, say $i=2$. The mapping

$$G=(g_1,g_2,\dots,g_n)=(f_2,f_1,f_3,\dots,f_n)$$

is a locally biholomorphic mapping in $\mathbb{P}^n$ with $\mathcal{S}G=\mathcal{S}F$. Its second component g_2 satisfies $g_2(a)=(\partial g_2/\partial z_1)(a)=0$. It follows now from (3.3) and successive derivatives with respect to z_1 that

$$\frac{\partial^m g_2}{\partial z_1^m}(a)=0 \quad , \; m\ge 2\,.$$

From the Taylor expansion of $\partial g_2/\partial z_1$ at $z=a$ in the variable z_1 we conclude that $(\partial g_2/\partial z_1)(z_1,a_2,\dots,a_n)\equiv 0$, and in particular, that $0=(\partial g_2/\partial z_1)(0,a_2,\dots,a_n)=(\partial f_1/\partial z_1)(0,a_2,\dots,a_n)$. This finishes the proof.

□

Before embarking into the analysis of univalence, we establish a final lemma in this section.

Lemma 3.4. *Let $F=(f_1,\dots,f_n):\mathbb{P}^n\to\mathbb{C}^n$ satisfy $||\mathcal{S}F||\le\alpha$, and suppose that $F(\xi)=0$ and $DF(\xi)=I$ at some $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{P}^n$. Then for each i,j there exist holomorphic function λ_{ij},μ_{ij} with the properties*

(i) $f_i=\lambda_{ij}f_j+\mu_{ij}$,

(ii) $\partial\lambda_{ij}/\partial z_j=\partial\mu_{ij}/\partial z_j\equiv 0$.

Proof. In differentiating $f_i u_0 = u_i$ we have that

$$\frac{\partial^2 f_i}{\partial z_j^2} u_0 + 2\frac{\partial f_i}{\partial z_j}\frac{\partial u_0}{\partial z_j} = S_{jj}^j F \frac{\partial f_i}{\partial z_j} u_0 \,,$$

thus

$$\frac{\partial^2 f_i}{\partial z_j^2} = \frac{\partial f_i}{\partial z_j}\left(S_{jj}^j F - 2\frac{\partial \log(u_0)}{\partial z_j}\right) \,,$$

By taking $j = i$ we obtain

$$\frac{\partial^2 f_j}{\partial z_j^2} = \frac{\partial f_j}{\partial z_j}\left(S_{jj}^j F - 2\frac{\partial \log(u_0)}{\partial z_j}\right) \,,$$

and after dividing the last equation by $\partial f_j/\partial z_j$, we obtain

$$\frac{\partial^2 f_j/\partial z_j^2}{\partial f_j/\partial z_j} = S_{jj}^j F - 2\frac{\partial \log(u_0)}{\partial z_j} \,.$$

Thus, for each i and j we have

$$\frac{\partial^2 f_i}{\partial z_j^2}\frac{\partial f_j}{\partial z_j} = \frac{\partial^2 f_j}{\partial z_j^2}\frac{\partial f_i}{\partial z_j} \,. \tag{3.7}$$

From this,

$$\frac{\partial}{\partial z_j}\left(\frac{\partial f_i/\partial z_j}{\partial f_j/\partial z_j}\right) \equiv 0 \,, \tag{3.8}$$

hence there exist holomorphic functions λ_{ij}, μ_{ij} independent of the variable z_j such that

$$f_i = \lambda_{ij} f_j + \mu_{ij} \,. \tag{3.9}$$

□

4. A Univalence Criterion

Our main result is

Theorem 4.1. *Let $F : \mathbb{P}^n \to \mathbb{C}^n$, $n \geq 2$, be a locally biholomorphic mapping such that*

$$||\mathcal{S}F|| \leq \frac{1}{3\sqrt{2}} \,.$$

Then F is univalent in $\mathbb{P}^n$.

Proof. Suppose F is not univalent. Then $F(\xi) = F(\zeta)$ for points $\xi \neq \zeta$ in $\mathbb{P}^n$. Hence $\xi_i \neq \zeta_i$ for some i, say for $i = 1$. Since an affine change leaves the Schwarzian invariant, we may assume that $F(\xi) = (0, \ldots, 0)$ and that $DF(\xi) = I$. Then for $i \neq n$ we have

$$f_i = \lambda_{in} f_n + \mu_{in} \,, \tag{4.1}$$

where λ_{in}, μ_{in} are independent of the variable z_n. Therefore

$$\frac{\partial f_i}{\partial z_n}(z) = \lambda_{in}(\hat{z}_n)\frac{\partial f_n}{\partial z_n}(z)\,, \quad i = 1, \ldots, n-1\,,$$

where $\hat{z}_n$ represents the point $(z_1, \ldots, z_{n-1})$. By evaluating $z = \xi$ we obtain that $\lambda_j(\hat{\xi}_n) = 0$. In addition, $f_i(\xi) = f_i(w) = 0$ implies that $\mu_{in}(\hat{\xi}_n) = \mu_{in}(\hat{\zeta}_n) = 0$, and using (3.6) we have that

$$f_i(\xi_1, \ldots, \xi_n) = f_i(\zeta_1, \ldots, \zeta_{n-1}, \xi_n) = 0\,, \quad i = 1, \ldots, n-1\,. \tag{4.2}$$

We repeat this process and write

$$f_i(z_1, \ldots, z_{n-1}, \xi_n) = \lambda_{i(n-1)}(z_1, \ldots, z_{n-2}, \xi_n) f_{n-1}(z_1, \ldots, z_{n-1}, \xi_n) + \mu_{i(n-1)}(z_1, \ldots, z_{n-2}, \xi_n)\,, \tag{4.3}$$

and find in analogous form that

$$f_i(\xi_1, \ldots, \xi_n) = f_i(\zeta_1, \ldots, \zeta_{n-2}, \xi_{n-1}, \xi_n) = 0\,, \quad i = 1, \ldots, n-2\,.$$

By iterating we finally obtain that the analytic function

$$f(w) = f_1(w, \xi_2, \ldots, \xi_n)\,, \quad z \in \mathbb{D}\,, \tag{4.4}$$

has $f(\xi_1) = f(\zeta_1) = 0$. Because of Lemma 3.3, this function is locally injective.

In order to obtain information about this function we will consider the Schwarzians

$$S_{11}^1 F = \frac{1}{\Delta}\sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial z_1^2}(-1)^{k+1} A_{k1} - \frac{2}{n+1}\frac{\partial}{\partial z_1}\log(\Delta)\,, \tag{4.5}$$

and

$$S_{12}^2 F = \frac{1}{\Delta}\sum_{k=1}^{n} \frac{\partial^2 f_k}{\partial z_1 \partial z_2}(-1)^{k} A_{k2} - \frac{1}{n+1}\frac{\partial}{\partial z_1}\log(\Delta)\,, \tag{4.6}$$

where A_{kj} is the determinant of differential DF resulting from eliminating of column j and row k. Because of the representation (3.5) we have that at points of the form $z = (w, \xi_2, \ldots, \xi_n)$, the differential DF is diagonal, hence $A_{kj} \neq 0$ only when $k = j$. Therefore, at such points,

$$S_{11}^1 F - 2S_{12}^2 F = \frac{1}{\Delta}\left(\frac{\partial^2 f_1}{\partial z_1^2} A_{11} - 2\frac{\partial^2 f_2}{\partial z_1 \partial z_2} A_{22}\right)\,. \tag{4.7}$$

Suppose for a moment that

$$\frac{\partial^2 f_2}{\partial z_1 \partial z_2}(\xi) = 0\,.$$

It follows from equation (3.5) that for $m \geq 2$

$$\frac{\partial^m f_2}{\partial z_1^{m-1} \partial z_2}(\xi) = 0\,, \tag{4.8}$$

which implies that

$$\frac{\partial^2 f_2}{\partial z_1 \partial z_2}(w, \xi_2, \dots, \xi_n) \equiv 0\,.$$

Hence, at $z = (w, \xi_2, \dots, \xi_n)$ we have that

$$S^1_{11}F - 2S^2_{12}F = \frac{f''}{f'}(w)\,,$$

and we obtain from Lemma 3.1 that

$$\left|\frac{f''}{f'}\right| \leq |S^1_{11}| + 2|S^2_{12}| \leq \frac{1}{1-|w|^2}\,.$$

The Becker criterion implies that $f = f(w)$ is injective, leading to a contradiction.

In general, when $\dfrac{\partial^2 f_2}{\partial z_1 \partial z_2}(\xi) \neq 0$ we consider a Möbius transformation

$$G = (g_1, \dots, g_n) = T \circ F = \left(\frac{f_1}{1+af_1}\,, \frac{f_2}{1+af_1}\,, \dots, \frac{f_n}{1+af_1}\right)\,,$$

for an appropriate value of a that makes

$$\frac{\partial^2 g_2}{\partial z_1 \partial z_2}(\xi) = 0\,.$$

Using that $F(\xi) = (0, \dots, 0)$, $DF(\xi) = I$, a simple calculation shows that

$$a = \frac{\partial^2 f_2}{\partial z_1 \partial z_2}(\xi)\,.$$

The mapping $G = (g_1, \dots, g_n)$ has $S^k_{ij}G = S^k_{ij}F$ for all i, j, k but becomes singular in $\mathbb{P}^n$ at points where $af_1 = -1$. Nevertheless, G is regular and locally biholomorphic in a subpolydisk $\mathbb{P}^n_\xi(r) = \{z \in \mathbb{C}^n : |z_i - \xi_i| < r\}$ becuase $f_1(\xi) = 0$. Note that $G(\xi) = (0, \dots, 0)$ and $DG(\xi) = I$. The proof of Lemmas 3.2, 3.3 and 3.4 show they remain valid for the mapping G in $\mathbb{P}^n_\xi(r)$. Equation (4.8) holds now for g_2, which implies that the function

$$\frac{\partial^2 f_2}{\partial z_1 \partial z_2}(w, \xi_2, \dots, \xi_n)$$

is holomorphic and identically zero for all $|w| < 1$. Because $S^1_{11}F - 2S^2_{12}F = S^1_{11}G - 2S^2_{12}G$, we conclude from (4.7) that g_1''/g_1' remains holomorphic for all $|w| <|$ and that g_1 satisfies the Becker univalence condition. Hence $f_1 = g_1/(1-ag_1)$ is again injective, a contradiction. This finishes the proof.

□

5. Examples

We present in this section several examples in all dimensions; for $n=2$ our main result is close to optimal.

Example 1: Let $F(z,w)=(f(z),w)$ be a locally univalent mapping defined in $\mathbb{P}^2$. A direct computation gives

$$\mathbb{S}^1 F=\frac{1}{3}\begin{pmatrix} -f''/f' & 0 \\ 0 & 0 \end{pmatrix}\,,\quad \mathbb{S}^2 F=\frac{1}{3}\begin{pmatrix} 0 & f''/f' \\ f''/f' & 0 \end{pmatrix}.$$

From this it follows that

$$\mathcal{S}F(z,w)(\vec{v},\vec{v})=\left(-\frac{1}{3}\frac{f''}{f'}\,v_1^2\,,\,\frac{2}{3}\frac{f''}{f'}\,v_1v_2\right)\,,$$

where $\vec{v}=(v_1,v_2)$ is a unitary vector in the Bergman norm, that is

$$\|\vec{v}\|^2=2\left(\frac{|v_1|^2}{(1-|z|^2)^2}+\frac{|v_2|^2}{(1-|w|^2)^2}\right)=1\,.$$

Then

$$\|\mathcal{S}F(z,w)(\vec{v},\vec{v})\|^2=\frac{2}{9}\left|\frac{f''}{f'}(z)\right|^2|v_1|^2\left(\frac{|v_1|^2}{(1-|z|^2)^2}+4\frac{|v_2|^2}{(1-|w|^2)^2}\right)\,,$$

a quantity that is maximized under the restriction on the norm of $\vec{v}$ when

$$\frac{|v_1|^2}{(1-|z|^2)^2}=\frac{1}{3}\quad,\quad\frac{|v_2|^2}{(1-|w|^2)^2}=\frac{1}{6}\,.$$

With this

$$\|\mathcal{S}F(z,w)|=\frac{\sqrt{2}}{3\sqrt{3}}\left|\frac{f''}{f'}(z)\right|(1-|z|^2)\,,$$

so that

$$||\mathcal{S}F||=\frac{\sqrt{2}}{3\sqrt{3}}\sup_{|z|<1}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|\,.$$

Hence F satisfies hypothesis of Theorem 4.1 precisely when

$$(1-|z|^2)\left|\frac{f''}{f'}(z)\right|\leq\frac{\sqrt{3}}{2}=0.866\,,$$

which is close to the optimal bound 1 for the univalence of f given by Becker's condition.

There are many univalent functions in the polydisk that have unbounded $||\mathcal{S}F||$. In fact, let $F(z,w)$ the Ropper-Suffridge extension defined by

$$F(z,w)=\left(f(z),\sqrt{f'(z)}\,w\right)\,,$$

where f is a univalent in $\mathbb{D}$ and the root is defined via branch of $\log f'(z)$ for which $\log f'(0) = 0$. Then F is univalent in $\mathbb{P}^2$, but a calculation shows that

$$\mathbb{S}^1 F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad \mathbb{S}^2 F = \begin{pmatrix} \frac{w}{2} Sf(z) & 0 \\ 0 & 0 \end{pmatrix} ,$$

thus $SF(z,w)(\vec{v},\vec{v}) = \left(0\,,\, \frac{w}{2} Sf(z) v_1^2\right)$, and so

$$\|SF(z,w)\| = \frac{1}{2\sqrt{2}} \frac{|w||Sf(z)|(1-|z|^2)^2}{(1-|w|^2)} \to \infty \,,$$

when $|w| \to 1$.

Example 2: Let $F(z) = (f(z_1), z_2, \ldots, z_n)$, with $f' \neq 0$ in $\mathbb{D}$. Hence $\Delta = f' \neq 0$ in $\mathbb{P}^n$. Then

$$S^{\,1}_{11} F(z) = \frac{n-1}{n+1} \frac{f''}{f'}(z_1) \,,$$

and

$$S^{\,i}_{1i} F(z) = -\frac{1}{n+1} \frac{f''}{f'}(z_1) \,, \quad i = 2, \ldots, n \,.$$

It is not difficult to see that $S^k_{ij} F \equiv 0$ when $k \neq i, j$. Hence,

$$SF(z)(\vec{v},\vec{v}) = \left(\frac{n-1}{n+1} \frac{f''}{f'}(z_1) v_1^2 \,,\, -\frac{2}{n+1} \frac{f''}{f'}(z_1) v_1 v_2 \,,\, \ldots \,,\, -\frac{2}{n+1} \frac{f''}{f'}(z_1) v_1 v_n \right) ,$$

where

$$\|\vec{v}\|^2 = 2 \left(\frac{|v_1|^2}{(1-|z_1|^2)^2} + \cdots + \frac{|v_n|^2}{(1-|z_n|^2)^2} \right) = 1 \,.$$

It follows that

$$\begin{aligned} \|SF(z)(\vec{v},\vec{v})\|^2 &= \frac{2}{(n+1)^2} \left| \frac{f''}{f'}(z_1) \right|^2 \left((n-1)^2 \frac{|v_1|^4}{(1-|z_1|^2)^2} + \frac{4|v_1 v_2|^2}{(1-|z_2|^2)^2} + \cdots + \frac{4|v_1 v_n|^2}{(1-|z_n|^2)^2} \right) \\ &= \frac{2}{(n+1)^2} \left| \frac{f''}{f'}(z_1) \right|^2 \left((n-1)^2 \frac{|v_1|^4}{(1-|z_1|^2)^2} + 2|v_1|^2 - 4 \frac{|v_1|^4}{(1-|z_1|^2)^2} \right) \\ &= \frac{2}{(n+1)^2} \left| \frac{f''}{f'}(z_1) \right|^2 \left((n^2-2n-3) \frac{|v_1|^4}{(1-|z_1|^2)^2} + 2|v_1|^2 \right) . \end{aligned}$$

If $n \geq 3$ we have that $n^2 - 2n - 3 \geq 0$, so that the maximal value of the left hand side occurs now when $v_2 = \cdots = v_n = 0$ and $2|v_1|^2 = (1-|z_1|^2)^2$. We conclude that

$$\|SF(z)\| = \frac{1}{\sqrt{2}} \left(\frac{n-1}{n+1} \right) (1-|z_1|^2) \left| \frac{f''}{f'}(z_1) \right| .$$

Thus F will satisfy the criterion in Theorem 4.1 when

$$(1-|z|^2)\left|\frac{f''}{f'}(z)\right| \leq \frac{1}{3}\left(\frac{n+1}{n-1}\right) .$$

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